Subjoined is a Table giving the dates of the various observations, with the reference numbers corresponding to those on the diagram, and with remarks on the state of the sky.

Number in diagram.	Date of observation.	Remarks.
I.	April 4th	
II.	January 8th	No mention of cloud.
III.	April 8th	
IV.	January 9th	No mention of cloud.
V.	March 8th.........	Extremely clear sky.
VI.	April 9th	No mention of cloud.
VII.	January 10th ...	Sky not good; thin hazy clouds, followed later in the night by a halo.
VIII.	February 9th......	
IX.	January 11th......	Much wind.
X.	February 10th ...	No mention of clouds.
XI.	January 12th......	Occasional small clouds, and rather hazy.
XII.	November 19th...	Clouds producing prismatic colours round the moon.
XIII.	March 13th	
XIV.	April 13th.........	Sky not good; fleecy clouds.
XV.	April 14th.........	Bad night; stopped after 10 minutes, in consequence of clouds.
XVI.	April 15th	Sky very clear.
XVII.	January 16th......	
XVIII.	September 20th...	Occasional clouds.
XIX.	February 16th ...	Sky hazy at sunset; occasional clouds.
XX.	April 16th.........	Sky apparently not quite so clear as on the preceding night.
XXI.	April 17th.........	
XXII.	November 22nd...	Fog and white frost, afterwards drift.
XXIII.	November 23rd...	No remark about cloud.

XVI. "On Linear Differential Equations."—No. III. By W. H. L. Russell, F.R.S.

Received June 11, 1870.

The integrals obtained in my last paper on this subject were deduced by the same process which afforded the determinants in the first paper. It is obvious that these integrals could be found by a more direct investigation. This is what I am now going to attempt. It will be found moreover that the present method will have the advantage of clearing away the ambiguities arising from the existence of common factors in the algebraical coefficient of the highest differential, and the denominator of the exponential in the solution. It will also be found to lead us to certain ulterior results.

Let us take the differential equation

$$(\alpha+\beta x)\frac{d^3y}{dx^3}+(\alpha'+\beta'x+\gamma'x^2)\frac{d^2y}{dx^2}+(\alpha''+\beta''x+\gamma''x^2)\frac{dy}{dx}$$
$$+(\alpha'''+\beta'''x+\gamma'''x^2)y=0.$$

Let us now put in this equation

$$y=\mathrm{E}(x)\epsilon^{\int dx(\mu+\nu x)}.$$

We shall easily see that it is impossible for the exponential to contain

higher powers of (x) than here given. Then we shall have

$$\frac{dy}{dx}=\left\{\frac{d\text{E}}{dx}+(\mu+\nu x)\text{E}\right\}\epsilon^{\int dx(\mu+\nu x)},$$

$$\frac{d^2y}{dx^2}=\left\{\frac{d^2\text{E}}{dx^2}+2(\mu+\nu x)\frac{d\text{E}}{dx}+(\mu+\nu x)^2\text{E}+\nu\text{E}\right\}\epsilon^{\int dx(\mu+\nu x)},$$

$$\frac{d^3y}{dx^3}=\left\{\frac{d^3\text{E}}{dx^3}+3(\mu+\nu x)\frac{d^2\text{E}}{dx^2}+3(\mu+\nu x)^2\frac{d\text{E}}{dx}+(\mu+\nu x)^3\text{E}\right.$$
$$\left.+3\nu\frac{d\text{E}}{dx}+3\nu(\mu+\nu x)\text{E}\right\}\epsilon^{\int dx(\mu+\nu x)}.$$

Substituting these values in the differential equation, and equating the coefficients of the highest powers of (x) to zero, we have

$$\nu^3\beta+\nu^2\gamma'=0,$$

whence

$$\nu=-\frac{\gamma'}{\beta},$$

and also

$$\nu^3\alpha+3\nu^2\mu\beta+2\mu\nu\gamma'+\nu^2\beta'+\gamma''\nu=0\,;$$

whence substituting for (ν) and reducing, we have

$$\mu=\frac{\alpha\gamma'}{\beta}+\frac{\gamma''}{\gamma'}-\frac{\beta'}{\beta},$$

as before.

The other integrals given in my last paper may be deduced in a similar way.

This method suggested to me that it was possible to ascertain if any linear differential equation admitted of a solution of the form $y=\frac{\text{P}}{\text{Q}}\epsilon^{\frac{p}{q}}$, where P, Q, p, q are rational and entire functions of (x).

Let, as before, the differential equation be

$$(\alpha_0+\alpha_1x+\ldots+\alpha_m x^m)\frac{d^ny}{dx^n}+(\beta_0+\beta_1x+\ldots+\beta_m x^m)\frac{d^{n-1}y}{dx^{n-1}}+\ldots=0.$$

Then it is easily seen that the factors of q must be divisors of

$$\alpha_0+\alpha_1x_1+\ldots+\alpha_m x^m\,;$$

hence if we have

$$\alpha_0+\alpha_1x+\ldots+\alpha_m x^m=(x-a)^{r'}(x-b)^{s'}\ldots,$$

we must have

$$y=\frac{\text{P}}{\text{Q}}\epsilon^{\eta_0+\eta_1x+\eta_2x^2+\ldots+\eta_\mu x^\mu}+\frac{\text{A}_r}{(x-a)^r}+\frac{\text{A}_{r-1}}{(x-a)^{r-1}}+\ldots+\frac{\text{B}_s}{(x-b)^s}+\ldots$$

Now this series can evidently be written in the form

$$y=\text{R}(x)\epsilon^{\eta_0+\eta_1x+\ldots+\eta_\mu x^m},$$

where R(x) can be expanded in descending powers of (x). Hence if this value of y be substituted in the proposed differential equation, we may determine η_0, η_1, $\ldots\eta_\mu$ by the same process as before.

To determine A_r, $A^{r-1}, \ldots,$ let

$$z=\frac{1}{x-a}, \text{ or } x=a+\frac{1}{z}.$$

Then the solution of the resulting linear differential equation in (z) will be of the form

$$y=R_1(z)\epsilon^{A_0+A_1z+A_2z^2+\ldots A_r.z^r};$$

where $R_1(z)$ can be expanded in descending powers of (z), and therefore A_0, A_1, $A_r, \ldots$ be determined as before.

In the same way, by putting $x=b+\frac{1}{z}$, we may determine B_0, $B_1, \ldots$

In order to exhibit the accuracy of this reasoning, I will form some differential equations from given primitives, and then see if the above process will enable us to reproduce these primitives as solutions.

Let us take the primitive

$$y=(x+1)\epsilon^{x^2+3x+\frac{1}{x-1}}.$$

From this we may deduce the differential equation

$$(x^3-x^2-x+1)\frac{d^2y}{dx^2}-(x^4+2x^3-6x^2-3x+4)\frac{dy}{dx}$$
$$-(2x^5+x^4+4x^3-x^2-5x-4)y=0.$$

Let

$$y=R(x)\epsilon^{\int dx(\mu+\nu x)}.$$

If we use higher powers of (x) in the exponential, they will not give us a result.

Substituting in the differential equation the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ given in the earlier part of this paper, and equating the highest terms to zero, we have

$$\nu^2-\nu-2=0 \text{ and } 2\mu\nu-\nu^2-\mu-2\nu-1=0,$$

whence

$$\nu=2 \text{ and } \mu=3, \text{ or } \nu=-1, \text{ and } \mu=0,$$

and therefore

$$\int dx(\mu+\nu x)=x^2+3x, \text{ or } -\frac{x^2}{2}.$$

The divisors of the first term are $x-1$ and $x+1$.

Let $x=\frac{1}{z}+1$, and the differential equation becomes

$$(2z^7+\ldots)\frac{d^2y}{dz^2}-(2z^7+\ldots)\frac{dy}{dz}+(3z^5+\ldots)y=0,$$

which gives a solution of the form $y=R_1(z)\epsilon^z$, when $z=\frac{1}{x-1}$. If we put $x=\frac{1}{z}-1$, the differential equation will be of the form

$$(Az^8+\ldots)\frac{d^2y}{dz^2}+(Bz^7+\ldots)\frac{dy}{dz}+(Cz^5+\ldots)y=0,$$

in which the numerical coefficients are of no consequence, as the equation does manifestly not admit of an exponential solution. If, then, the differential equation admits of a solution in the proposed form, it must be one of the two forms,

$$y=\mathrm{R}(x)\epsilon^{x^2+3x+\frac{1}{x-1}} \text{ or } y=\mathrm{R}(x)\epsilon^{-\frac{x^2}{2}+\frac{1}{x-1}},$$

where $\mathrm{R}(x)$ is a rational and entire function of (x) or a rational fraction. Using the first form, we should of course determine it equal to $x+1$.

As a second example, we form from the primitive, $y=(x-1)\epsilon^{x+\frac{1}{x+1}}$, the equation

$$(x-1)(x+1)^2\frac{d^2y}{dx^2}+(2x^2+x-3)\frac{dy}{dx}-(x^3+5x^2+4x+1)y=0.$$

Here we must put $y=\mathrm{R}(x)\epsilon^{\int\mu dx}$, higher powers of (x) in the exponential leading to no result. Substituting, we find $\mu=\pm1$. Let $x=\frac{1}{z}-1$, and the differential equation becomes

$$(2z^5+\ldots)\frac{d^2y}{dz^2}+(2z^5+\ldots)\frac{dy}{dz}+(z^3+\ldots)y=0,$$

which gives $y=\mathrm{R}(z)\epsilon^{z}$. If we put $x=\frac{1}{z}$, we find no exponential solution.

Consequently the solution of the equation, if it can be obtained under the form we are now considering, must be one of the two expressions,

$$y=\mathrm{R}(x)\epsilon^{x+\frac{1}{x-1}} \text{ and } y=\mathrm{R}(x)\epsilon^{-x+\frac{1}{x-1}}.$$

As a third example, I take the primitive,

$$y=x\epsilon^{x+\frac{1}{x^2}},$$

and from it the differential equation

$$x^3\frac{d^2y}{dx^2}+(2x^2+2)\frac{dy}{dx}-(x^3+4x^2+2x-2)y=0.$$

We must evidently here put $y=\mathrm{R}(x)\epsilon^{\int\mu dx}$, which gives $\mu=\pm1$. If $x=\frac{1}{z}$, the equation becomes

$$z^4\frac{d^2y}{dz^2}-2z^5\frac{dy}{dz}-(1+4z+2z^2-2z^3)y=0.$$

If we put here $y=\mathrm{R}(z)\epsilon^{\int dz(\mu+\nu z)}$, employ the formulæ given in the first part of the paper, and equate the coefficients of the highest powers of (z) to zero, we have $\nu^2-2\nu=0$, $2\mu\nu-2\mu=0$, whence $\nu=2$, and $\mu=0$; and y must be one of the two forms $\mathrm{R}(x)\epsilon^{x+\frac{1}{x^2}}$, or $\mathrm{R}(x)\epsilon^{-x+\frac{1}{x^2}}$.